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MULTIVARIABLE ASKEY-WILSON POLYNOMIALS AND QUANTUM COMPLEX GRASSMANNIANS

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ABSTRACT. We present a one-parameter family of constant solutions of the reflection equation and define a family of quantum complex Grassmannians endowed with a transitive action of the quantum unitary group. By computing the radial part of a suitable Casimir operator, we identify the zonal spherical functions (i.e. infinitesimally bi-invariant matrix coefficients of finite-dimensional irreducible representations) as multivariable Askey-Wilson polynomials containing two continuous and two discrete parameters.

0. Introduction

Every classical compact symmetric space M can be realized as an orbit of square matrices, the action of the symmetry group G being given by $X \mapsto TXT^t$ or $X \mapsto TXT^*$ ($X \in M, T \in G$). This approach has proved to be fruitful in the study of quantum symmetric spaces as well. The proper way to quantize the space of square matrices endowed with either of the above-mentioned group actions would be to make use of commutation relations given by one of two types of so-called *reflection equations*. These equations pop up at various places in mathematics, notably in the theory of quantum integrable systems (cf. [KU]). The quantized symmetric space M_q is then realized as the quantum orbit containing a certain "classical" or \mathbb{C} -valued point in the quantum space of square matrices (i.e. a \mathbb{C} -valued evaluation homomorphism on the quantized algebra of functions). These "classical" points are naturally in 1-1 correspondence with constant solutions J of the reflection equation.

There is another, slightly different, but closely related, way to explain the role of the reflection equation in the theory of quantum symmetric spaces. Given any

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square matrix J with complex coefficients, there is a straightforward way to define a two-sided coideal \mathfrak{k}_J in the quantized universal enveloping algebra using the L -operators introduced in [RTF]. The reflection equation now more or less guarantees the existence of sufficiently many \mathfrak{k}_J -fixed vectors. This fact is obviously central to the study of zonal spherical functions. If the matrix J satisfies the reflection equation, the invariant ring corresponding to the coideal \mathfrak{k}_J defines a quantum homogeneous G_q -space that is G_q -isomorphic to the quantum orbit passing through the “classical” point associated with J .

In [N], the first author used the ideas sketched above to construct and analyse quantum analogues of the symmetric spaces $GL(n)/SO(n)$ and $GL(2n)/Sp(n)$. The corresponding zonal spherical functions were shown to be expressible in terms of Macdonald’s symmetric polynomials corresponding to root system A_{n-1} (cf. [M]). As a follow-up, the first and third author extended this approach to all classical compact symmetric spaces ([NS]). In all these cases, the zonal spherical functions could be expressed as Macdonald’s polynomials ([M]) or Koornwinder’s multivariable Askey-Wilson polynomials ([KW2]), depending on the restricted root system of the symmetric space.

The quantum symmetric spaces treated in [N], [NS], however, do not exhaust all known examples of quantum analogues of classical compact symmetric spaces. In the early stages of quantum group theory, Podleś had already defined a continuously parametrized (parameter different from q !) family of mutually non-isomorphic $SU_q(2)$ -homogeneous spaces. Each of these spaces can be regarded as a quantum analogue of the classical 2-sphere. The zonal spherical functions on these quantum spheres were analysed by Mimachi and the first author ([NM]), and by [KW1]. They can be expressed as a certain subfamily of Askey-Wilson polynomials ([AW]) or limit cases thereof.

Recently, the first and second authors generalized these results to quantum complex projective spaces of arbitrary dimension ([DN]). They studied a family of quantum projective spaces which were introduced in [KV] and depend on a continuous parameter. The zonal spherical functions were expressed as Askey-Wilson polynomials depending on two continuous and one discrete parameter or limit cases thereof. See [Dz] for a more extensive discussion of the relation between these parametrized families of quantum symmetric spaces and the examples treated in [N], [NS].

In this paper, we announce some results which constitute a generalization of [DN]. We present a one-parameter family of constant solutions to one type of reflection equation, and use them to define a family of quantum homogeneous $U_q(n)$ -spaces that can be regarded as a quantum analogue of a complex Grassmannian of arbitrary rank. It turns out that the zonal spherical functions on these quantum Grassmannian spaces are expressed as a subfamily of Koornwinder’s multivariable Askey-Wilson polynomials depending on two continuous and two discrete parameters. This last result is proved by computing the radial part of a suitable Casimir operator in the quantized universal enveloping algebra, a method which was first used by Koornwinder ([KW1]) in the $SU_q(2)$ -case, and subsequently generalized by the first author ([N]) to higher rank quantum symmetric spaces.

This paper does not contain any proofs. Full proofs and more details will be given elsewhere.

1. Recall on multivariable Askey-Wilson polynomials

We recall the definition of multivariable Askey-Wilson polynomials ([KW2]). Our notation is slightly different from [KW2].

Let $P_\Sigma := \bigoplus_{1 \leq k \leq l} \mathbb{Z}\varepsilon_k$ denote the weight lattice of the root system BC_l , and P_Σ^+ the cone of dominant weights in P_Σ . A dominant weight $\lambda = \sum_{k=1}^l \lambda_k \varepsilon_k \in P_\Sigma$ is then characterized by the condition $\lambda_1 \geq \dots \geq \lambda_l \geq 0$. Let \leq denote the usual dominance ordering on weights. Recall that $\lambda \leq \mu$ if and only if $\sum_{i=1}^k \lambda_i \leq \sum_{i=1}^k \mu_i$ for all $1 \leq k \leq l$.

We will use the notation $x_k := e^{\varepsilon_k}$ ($1 \leq k \leq \ell$) to refer to the formal exponentials. Let $\mathbb{C}[x^{\pm 1}] := \mathbb{C}[x_1^{\pm 1}, \dots, x_l^{\pm 1}]$ denote the algebra of Laurent polynomials in the variables x_k ($1 \leq k \leq l$). $\mathbb{C}[x^{\pm 1}]$ can be viewed as the group algebra of the weight lattice P_Σ . The Weyl group $W := \mathbb{Z}_2^l \rtimes \mathfrak{S}_l$ of BC_l acts on $\mathbb{C}[x^{\pm 1}]$ in a natural way by permutations and sign changes of the x_k . Let $\mathbb{C}[x^{\pm 1}]^W \subset \mathbb{C}[x^{\pm 1}]$ denote the subalgebra of W -invariant elements.

Let $0 < q < 1$. Write $T_{q, x_k}: \mathbb{C}[x^{\pm 1}] \rightarrow \mathbb{C}[x^{\pm 1}]$ for the q -shift operator sending x_k to qx_k . Koornwinder's q -difference operator $D_{\varepsilon_1}: \mathbb{C}[x^{\pm 1}] \rightarrow \mathbb{C}[x^{\pm 1}]$ is defined by

$$D_{\varepsilon_1} := \sum_{k=1}^l (\Phi_k^+(x) T_{q, x_k} + \Phi_k^-(x) T_{q, x_k}^{-1}) - \Phi^0(x), \quad (1.1)$$

where

$$\begin{aligned} \Phi_k^+(x) &:= \frac{(1 - ax_k)(1 - bx_k)(1 - cx_k)(1 - dx_k)}{(1 - x_k^2)(1 - qx_k^2)} \prod_{i \neq k} \frac{(tx_k - x_i)(tx_k x_i - 1)}{(x_k - x_i)(x_k x_i - 1)}, \\ \Phi_k^-(x) &:= \frac{(x_k - a)(x_k - b)(x_k - c)(x_k - d)}{(x_k^2 - 1)(x_k^2 - q)} \prod_{i \neq k} \frac{(x_k - tx_i)(x_k x_i - t)}{(x_k - x_i)(x_k x_i - 1)}, \\ \Phi^0(x) &:= \sum_{k=1}^l \Phi_k^+(x) + \Phi_k^-(x). \end{aligned} \quad (1.2)$$

Here we assume that the parameters a, b, c, d, t are complex numbers satisfying

$$0 < t < 1, \quad -q \leq abcd < 1. \quad (1.3)$$

The operator D_{ε_1} maps $\mathbb{C}[x^{\pm 1}]^W$ into itself.

It can be shown that there is a unique family $(P_\lambda)_{\lambda \in P_\Sigma^+}$ in $\mathbb{C}[x^{\pm 1}]^W$ such that

$$\begin{aligned} \text{(i)} \quad P_\lambda &= m_\lambda + \sum_{\mu < \lambda} c_{\lambda\mu} m_\mu, \\ \text{(ii)} \quad D_{\varepsilon_1} P_\lambda &= c_{\lambda\lambda} P_\lambda. \end{aligned} \quad (1.4)$$

Here $m_\lambda := \sum_{\mu \in W\lambda} x^\mu$ is the orbit sum corresponding to $\lambda \in P_\Sigma^+$. One has

$$c_{\lambda\lambda} = \sum_{k=1}^l (q^{-1} abcd t^{2l-k-1} (q^{\lambda_k} - 1) + t^{k-1} (q^{-\lambda_k} - 1)) \quad (\lambda \in P_\Sigma^+). \quad (1.5)$$

Note that $c_{\lambda\lambda} \neq c_{\mu\mu}$ for $\lambda > \mu$, provided (1.3) is satisfied (cf. [KS], Proposition 4.6). The elements

$$P_\lambda = P_\lambda(x; a, b, c, d; q, t) \quad (1.6)$$

will be called *multivariable Askey-Wilson polynomials* (associated with the root system BC_l). For certain values of the parameters one reobtains Macdonald's polynomials ([M]) corresponding to the pairs (BC_l, B_l) or (BC_l, C_l) .

Assume now for simplicity that $0 < t < 1$ and that a, b, c, d are real numbers with $|a|, |b|, |c|, |d| < 1$. Let $T_\Sigma^\mathbb{C} := (\mathbb{C}^*)^l$ denote the complex torus of dimension l . We denote its natural compact real form by T_Σ . Elements of $\mathbb{C}[x^{\pm 1}]$ can be naturally viewed as polynomial functions on $T_\Sigma^\mathbb{C}$.

Recall the notation of *q-shifted factorials*:

$$(a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k), \quad (a_1, \dots, a_s; q)_n := \prod_{j=1}^s (a_j; q)_n, \quad (a; q)_\infty := \lim_{n \rightarrow \infty} (a; q)_n. \quad (1.7)$$

Under the above-mentioned conditions on a, b, c, d, t , the infinite product

$$\Delta^+ := \prod_{i=1}^l \frac{(x_i^2; q)_\infty}{(ax_i, bx_i, cx_i, dx_i; q)_\infty} \cdot \prod_{1 \leq i < j \leq l} \frac{(x_i/x_j; q)_\infty (x_i x_j; q)_\infty}{(tx_i/x_j; q)_\infty (tx_i x_j; q)_\infty} \quad (1.8)$$

defines a continuous function on T_Σ . If we put

$$\Delta(x) := \Delta^+(x) \Delta^+(x^{-1}), \quad (1.9)$$

then Δ is a positive continuous function on T_Σ . The Askey-Wilson polynomials $P_\lambda(x; a, b, c, d; q, t)$ are mutually orthogonal with respect to the weight function Δ :

$$\int_{T_\Sigma} P_\lambda(x) P_\mu(x^{-1}) \Delta(x) dx = 0, \quad \lambda \neq \mu. \quad (1.10)$$

Here dx denotes the Haar measure on the real torus T_Σ .

For general values of the parameters, an explicit expression of the orthogonality measure was recently written down by Stokman [ST2].

2. Quantum complex Grassmannians

Let us fix some notation for the quantum unitary group (cf. [J], [RTF], [N]). More details can be found in [N]. Let $0 < q < 1$ and $n \geq 2$. Let $\mathcal{A}_q = \mathcal{A}_q(U(n))$ denote the algebra of functions on the *quantum unitary group* $U_q(n)$. The matrix elements of the vector representation V with basis $(v_i)_{1 \leq i \leq n}$ are written $t_{ij} \in \mathcal{A}_q$ ($1 \leq i, j \leq n$). They satisfy the commutation relations

$$RT_1 T_2 = T_2 T_1 R, \quad (2.1)$$

where $T := (t_{ij})_{1 \leq i, j \leq n}$ is an $n \times n$ matrix with coefficients in \mathcal{A}_q , $T_1 := T \otimes \text{id}$ and $T_2 := \text{id} \otimes T$ are Kronecker matrix products, and $R \in \text{End}(V \otimes V)$ is the invertible $n^2 \times n^2$ matrix defined by

$$R := \sum_{ij} q^{\delta_{ij}} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i > j} e_{ij} \otimes e_{ji}. \quad (2.2)$$

The $e_{ij} \in \text{End}(V)$ denote the standard matrix units with respect to the basis (v_i) . We put

$$R^+ := PRP, \quad R^- := R^{-1}, \quad (2.3)$$

where $P \in \text{End}(V \otimes V)$ is the usual permutation operator. The Hopf $*$ -algebra \mathcal{A}_q is spanned by the coefficients of its finite-dimensional unitary corepresentations (cf. [Wz], [NYM], [DK2]).

Let P denote the free \mathbb{Z} -module of rank n with canonical basis $(\varepsilon_i)_{1 \leq i \leq n}$. The quantized universal enveloping algebra $\mathcal{U}_q = \mathcal{U}_q(\mathfrak{gl}(n))$ is the algebra generated by the symbols q^h ($h \in P^*$) and e_i, f_i ($1 \leq i \leq n-1$) subject to the well-known quantized Weyl-Serre relations. The algebra \mathcal{U}_q is also generated by the so-called L -operators $L_{ij}^+, L_{ij}^- \in \mathcal{U}_q$ (cf. [J], [RTF], [N]). The matrices $L^\pm := \sum_{ij} e_{ij} \otimes L_{ij}^\pm$ with coefficients in \mathcal{U}_q satisfy the following relations:

$$R^+ L_1^\varepsilon L_2^\varepsilon = L_2^\varepsilon L_1^\varepsilon R^+ \quad (\varepsilon = \pm), \quad R^+ L_1^+ L_2^- = L_2^- L_1^+ R^+, \quad (2.4)$$

where $L_1^\pm := L^\pm \otimes \text{id}$ and $L_2^\pm := \text{id} \otimes L^\pm$ are Kronecker matrix products. The Hopf $*$ -algebra structure on \mathcal{U}_q is determined by:

$$\Delta(L_{ij}^\pm) = \sum_k L_{ik}^\pm \otimes L_{kj}^\pm, \quad \varepsilon(L_{ij}^\pm) = \delta_{ij}, \quad (L_{ij}^\pm)^* = S(L_{ji}^\mp), \quad (2.5)$$

for $1 \leq i, j \leq n$. We put $\tau = * \circ S: \mathcal{U}_q \rightarrow \mathcal{U}_q$. Then

$$\tau(L_{ij}^\pm) = L_{ji}^\mp \quad (1 \leq i, j \leq n). \quad (2.6)$$

Recall that a left \mathcal{U}_q -module W is called P -weighted if it has a vector space basis consisting of weight vectors with weights in P . The cone $P^+ \subset P$ of dominant weights consists by definition of all weights $\lambda = \sum_k \lambda_k \varepsilon_k \in P$ such that $\lambda_1 \geq \dots \geq \lambda_n$. There is a 1-1 correspondence $\lambda \longleftrightarrow V(\lambda)$ between dominant weights and irreducible P -weighted finite-dimensional left \mathcal{U}_q -modules such that $\lambda \in P^+$ is the highest weight of $V(\lambda)$ (cf. [Ro]). Recall that $\lambda \in P^+$ is called a highest weight of a left \mathcal{U}_q -module W if there exists a non-zero vector $w \in W$ such that $q^h \cdot w = q^{(h, \lambda)} w$ and $e_i \cdot w = 0$ for all $1 \leq i \leq n-1$.

Given a left \mathcal{U}_q -module W , we define a right \mathcal{U}_q -module structure on the same underlying vector space W by putting

$$w \cdot u := u^* \cdot w \quad (w \in W, u \in \mathcal{U}_q). \quad (2.7)$$

The corresponding right \mathcal{U}_q -module will be denoted by W° .

The Hopf $*$ -algebra pairing $\langle \cdot, \cdot \rangle$ between \mathcal{U}_q and \mathcal{A}_q is determined by

$$\langle L_1^\pm, T_2 \rangle = R^\pm, \quad \langle L^\pm, \det_q \rangle = q^{\pm 1} \text{id}. \quad (2.8)$$

This pairing is nondegenerate in the sense that the canonical mapping $\mathcal{A}_q \rightarrow \text{Hom}_{\mathbb{C}}(\mathcal{U}_q, \mathbb{C})$ is injective. Under this duality, the finite-dimensional corepresentations of \mathcal{A}_q are in 1-1 correspondence with the finite-dimensional P -weighted representations of \mathcal{U}_q .

Using the pairing $\langle \cdot, \cdot \rangle$ between \mathcal{U}_q and \mathcal{A}_q , one can identify \mathcal{A}_q with a subspace of the algebraic linear dual of \mathcal{U}_q . One defines a \mathcal{U}_q -bimodule structure on \mathcal{A}_q by putting:

$$u \cdot a := (\text{id} \otimes u) \circ \Delta(a), \quad a \cdot u := (u \otimes \text{id}) \circ \Delta(a) \quad (u \in \mathcal{U}_q, a \in \mathcal{A}_q). \quad (2.9)$$

The action of the L-operators is given by:

$$L_1^\varepsilon \cdot T_2 = T_2 R^\varepsilon, \quad T_2 \cdot L_1^\varepsilon = R^\varepsilon T_2 \quad (\varepsilon = \pm). \quad (2.10)$$

The multiplication $\mathcal{A}_q \otimes \mathcal{A}_q \rightarrow \mathcal{A}_q$ and the unit mapping $\mathbb{C} \rightarrow \mathcal{A}_q$ are \mathcal{U}_q -bimodule homomorphisms. In other words, \mathcal{A}_q is an *algebra with two-sided \mathcal{U}_q -symmetry*. One has the following decomposition of \mathcal{A}_q into irreducible \mathcal{U}_q -bimodules:

$$\mathcal{A}_q = \bigoplus_{\lambda \in P^+} V(\lambda) \otimes V(\lambda)^\circ. \quad (2.11)$$

Here the subspace $W(\lambda) := V(\lambda) \otimes V(\lambda)^\circ \subset \mathcal{A}_q$ is spanned by the matrix coefficients of the (co-)representation $V(\lambda)$. The decomposition (2.11) can also be characterized as the simultaneous eigenspace decomposition of \mathcal{A}_q with respect to the natural action of the center $\mathcal{Z}\mathcal{U}_q \subset \mathcal{U}_q$. Let $h: \mathcal{A}_q \rightarrow \mathbb{C}$ denote the Haar functional on \mathcal{A}_q (cf. [Wz], [DK2]). Then $\langle a, b \rangle := h(b^* a)$ defines a positive definite inner product on \mathcal{A}_q with respect to which the subspaces $W(\lambda) \subset \mathcal{A}_q$ are mutually orthogonal (Schur orthogonality).

We now proceed to define a family of quantum Grassmannians. Let X be an $n \times n$ matrix with coefficients in any ring. Consider the following *reflection equation*:

$$R_{12} X_1 R_{12}^{-1} X_2 = X_2 R_{21}^{-1} X_1 R_{21}. \quad (2.12)$$

Here $X_1 := X \otimes \text{id}$ and $X_2 := \text{id} \otimes X$ are Kronecker matrix products, $R_{12} = R$, $R_{21} = PRP$ (P being the permutation operator). Define the algebra $\mathcal{C}_q = \mathcal{C}_q(n)$ of functions on the quantum space of q -Hermitian matrices as the algebra generated by the symbols x_{ij} ($1 \leq i, j \leq n$) subject to the relations given by the reflection equation (2.12). There is a unique $*$ -operation on \mathcal{C}_q such that $x_{ij}^* = x_{ji}$. In shorthand notation, we write $X^* = X$.

Proposition 2.1 — *There is a unique $*$ -algebra homomorphism $\delta: \mathcal{C}_q \rightarrow \mathcal{A}_q \otimes \mathcal{C}_q$ such that*

$$\delta(x_{ij}) = \sum_{r,s} t_{ir} t_{js}^* \otimes x_{rs} \quad \text{or} \quad \delta(X) = T X T^*. \quad (2.13)$$

The mapping δ is a comodule mapping, hence defines an action of the quantum unitary group $U_q(n)$ on the quantum space of q -Hermitian matrices.

In the terminology of [DK1], a *classical point* in the quantum space of q -Hermitian matrices is a $*$ -algebra homomorphism $\tilde{\varepsilon}: \mathcal{C}_q \rightarrow \mathbb{C}$. By the definition of the algebra \mathcal{C}_q , such mappings $\tilde{\varepsilon}$ are in 1-1 correspondence with $n \times n$ matrices J with complex coefficients satisfying the reflection equation (2.12) and such that $J^* = J$. The correspondence is given by $\tilde{\varepsilon}(X) = J$.

In the remainder of this paper, we fix an integer $1 \leq l \leq [\frac{n}{2}]$. Let σ be a real parameter. Define an $n \times n$ matrix J^σ by putting

$$J^\sigma := \sum_{1 \leq k \leq l} q^\sigma (q^{-\sigma} - q^\sigma) e_{kk} + \sum_{l < k < l'} e_{kk} - \sum_{k \leq l \text{ or } k \geq l'} q^\sigma e_{kk'}, \quad (2.14)$$

where $k' := n + 1 - k$ ($1 \leq k \leq n$).

Proposition 2.2 — *The matrix J^σ satisfies the reflection equation (2.12) for any value of the parameter $\sigma \in \mathbb{R}$.*

Define a mapping $\Psi^\sigma: \mathcal{C}_q \rightarrow \mathcal{A}_q$ by putting $\Psi^\sigma := (\text{id} \otimes \tilde{\varepsilon}^\sigma) \circ \delta$. Then Ψ^σ is a $*$ -algebra homomorphism intertwining the natural (left) coactions of \mathcal{A}_q on \mathcal{C}_q and itself. We write $\mathcal{B}_q^\sigma := \Psi^\sigma(\mathcal{C}_q) \subset \mathcal{A}_q$ for the image of Ψ^σ . \mathcal{B}_q^σ is a $*$ -subalgebra and left coideal in \mathcal{A}_q . It can be viewed as the algebra of functions on the $U_q(n)$ -orbit of the classical point defined by J^σ in the quantum space of q -Hermitian matrices. The action of $U_q(n)$ on this quantum orbit is transitive by construction (cf. [DK1]). If $q = 1$ then the algebra \mathcal{B}_q^σ ($\sigma \in \mathbb{R}$) is $U(n)$ -isomorphic with the algebra of functions on the complex Grassmannian of rank l endowed with its natural $U(n)$ -action.

The next obvious step is to look for a stabilizer “subgroup”. Define an $n \times n$ matrix M^σ with coefficients in \mathcal{U}_q by putting

$$M^\sigma := L^+ J^\sigma - J^\sigma L^- \in \text{End}(V) \otimes \mathcal{U}_q. \quad (2.15)$$

The subspace $\mathfrak{k}^\sigma \subset \mathcal{U}_q$ is by definition spanned by the coefficients of the matrix M^σ .

Lemma 2.3 — *The subspace $\mathfrak{k}^\sigma \subset \mathcal{U}_q$ is a τ -invariant two-sided coideal.*

An element $a \in \mathcal{A}_q$ is called left \mathfrak{k}^σ -invariant if $\mathfrak{k}^\sigma \cdot a = 0$.

Proposition 2.4 — *The subalgebra $\mathcal{B}_q^\sigma \subset \mathcal{A}_q$ consists precisely of all left \mathfrak{k}^σ -invariant elements in \mathcal{A}_q .*

In other words, the two-sided coideal \mathfrak{k}^σ can be regarded as an infinitesimal stabilizer of the quantum orbit defined by the matrix J^σ . It can be shown that, in the limit $q \rightarrow 1$, the coideal $\mathfrak{k}^\sigma \subset \mathcal{U}_q$ tends to a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{C})$ which is conjugate to the Lie algebra \mathfrak{k} of the subgroup $U(l) \times U(n-l) \subset U(n)$.

Let W be a finite-dimensional left \mathcal{U}_q -module. A vector $w \in W$ is called \mathfrak{k}^σ -fixed if $\mathfrak{k}^\sigma \cdot w = 0$. The subspace of \mathfrak{k}^σ -fixed vectors in W is written $W_{\mathfrak{k}^\sigma}$. We have the following key lemma:

Lemma 2.5 — *Let V denote the vector representation of \mathcal{U}_q , V^* its contragredient with dual basis (v_i^*) . The element $w_{J^\sigma} := \sum_{i,j} J_{ij}^\sigma v_i \otimes v_j^* \in V \otimes V^*$ is a \mathfrak{k}^σ -fixed vector.*

Given any complex matrix J , one can define a matrix M_J with coefficients in \mathcal{U}_q as in (2.15), and a corresponding two-sided coideal $\mathfrak{k}_J \subset \mathcal{U}_q$. In order that the element $w_J := \sum_{i,j} J_{ij} v_i \otimes v_j^* \in V \otimes V^*$ (V vector representation) be a \mathfrak{k}_J -fixed vector it is necessary and sufficient for the matrix J to satisfy the reflection equation (2.12).

Let us identify elements $\lambda \in P^+$ with sequences $(\lambda_1, \dots, \lambda_n)$ of integers such that $\lambda_1 \geq \dots \geq \lambda_n$.

Theorem 2.6 — *For any $\lambda \in P^+$, the subspace $V(\lambda)_{\mathfrak{k}^\sigma}$ is at most one-dimensional. There are non-zero \mathfrak{k}^σ -fixed vectors in $V(\lambda)$ if and only if $\lambda \in P^+$ is of the form*

$$\lambda = (\mu_1, \dots, \mu_l, 0, \dots, 0, -\mu_l, \dots, -\mu_1), \quad \mu_1 \geq \dots \mu_l \geq 0. \quad (2.16)$$

A dominant weight $\lambda \in P^+$ and the corresponding representation $V(\lambda)$ are called *spherical* if λ is of the form (2.16). The set of spherical dominant weights will be denoted by $P_\mathfrak{k}^+$.

Corollary 2.7 — *One has the following irreducible decomposition of the right \mathcal{U}_q -module (or left \mathcal{A}_q -comodule) \mathcal{B}_q^σ :*

$$\mathcal{B}_q^\sigma = \bigoplus_{\lambda \in P_\mathfrak{k}^+} V(\lambda)^\circ. \quad (2.17)$$

3. Zonal spherical functions

Let σ, τ be real parameters. Define $\mathcal{H}^{(\sigma, \tau)} \subset \mathcal{A}_q$ as the subspace of left \mathfrak{k}^σ -invariant and right \mathfrak{k}^τ -invariant elements. It is a $*$ -subalgebra of \mathcal{A}_q .

Proposition 3.1 — *If we put $\mathcal{H}^{(\sigma, \tau)}(\lambda) := \mathcal{H}^{(\sigma, \tau)} \cap W(\lambda)$ ($\lambda \in P^+$) then we have the decomposition*

$$\mathcal{H}^{(\sigma, \tau)} = \bigoplus_{\lambda \in P_\mathfrak{k}^+} \mathcal{H}^{(\sigma, \tau)}(\lambda). \quad (3.1)$$

Each of the subspaces $\mathcal{H}^{(\sigma, \tau)}(\lambda)$ ($\lambda \in P_\mathfrak{k}^+$) is one-dimensional.

Any non-zero element of $\mathcal{H}^{(\sigma, \tau)}(\lambda)$ ($\lambda \in P_\mathfrak{k}^+$) is called a *zonal spherical function* corresponding to the spherical weight λ .

Let $\mathcal{A}(\mathbb{T}) := \mathbb{C}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ be the algebra of Laurent polynomials in the variables z_i ($1 \leq i \leq n$). There is a unique Hopf $*$ -algebra structure on $\mathcal{A}(\mathbb{T})$ such that

$$\Delta(z_i) = z_i \otimes z_i, \quad \varepsilon(z_i) = 1, \quad z_i^* = z_i^{-1} \quad (1 \leq i \leq n). \quad (3.2)$$

There is a unique surjective Hopf $*$ -algebra morphism

$$|_{\mathbb{T}}: \mathcal{A}_q \longrightarrow \mathcal{A}(\mathbb{T})$$

mapping $t_{ij} \in \mathcal{A}_q$ onto $\delta_{ij} z_i \in \mathcal{A}(\mathbb{T})$. The torus \mathbb{T} can be viewed as a subgroup of the quantum unitary group $U_q(n)$. The mapping $|_{\mathbb{T}}$ then is the corresponding restriction of functions. Let us introduce new variables x_i ($1 \leq i \leq l$) in the algebra $\mathcal{A}(\mathbb{T})$ by putting

$$x_1 := z_1 z_n^{-1}, \quad x_2 := z_2 z_{n-1}^{-1}, \quad \dots, \quad x_l := z_l z_{n+1-l}^{-1}. \quad (3.3)$$

Note that the Weyl group W of the root system BC_l acts naturally on the subalgebra of $\mathcal{A}(\mathbb{T})$ generated by the $x_i^{\pm 1}$. The subalgebra of W -invariant elements is written $\mathbb{C}[x^{\pm 1}]^W = \mathbb{C}[x_1^{\pm 1}, \dots, x_l^{\pm 1}]^W$.

Theorem 3.2 — *The mapping $|_{\mathbb{T}}: \mathcal{A}_q \longrightarrow \mathcal{A}(\mathbb{T})$ maps $\mathcal{H}^{(\sigma, \tau)}$ injectively onto its image $\mathcal{H}_{|_{\mathbb{T}}}^{(\sigma, \tau)} \subset \mathcal{A}(\mathbb{T})$. One has*

$$\mathcal{H}_{|_{\mathbb{T}}}^{(\sigma, \tau)} = \mathbb{C}[x_1^{\pm 1}, \dots, x_l^{\pm 1}]^W. \quad (3.4)$$

In particular, the algebra $\mathcal{H}^{(\sigma, \tau)}$ is commutative.

Consider the Casimir operator (cf. [RTF])

$$C := \sum_{ij} q^{2(n-i)} L_{ij}^+ S(L_{ij}^-) \in \mathcal{U}_q. \quad (3.5)$$

C is central in \mathcal{U}_q and acts as a scalar on each subspace $W(\lambda)$ ($\lambda \in P^+$). The corresponding eigenvalue is

$$\chi_\lambda(C) = \sum_{k=1}^n q^{2(\lambda_k + n - k)}. \quad (3.6)$$

Since C is central in \mathcal{U}_q , the left action of C on \mathcal{A}_q preserves the subalgebra $\mathcal{H}^{(\sigma, \tau)}$, hence acts as a scalar on each of the subspaces $\mathcal{H}^{(\sigma, \tau)}(\lambda)$ ($\lambda \in P_\Sigma^+$). Namely, each zonal spherical function $\varphi \in \mathcal{H}^{(\sigma, \tau)}(\lambda)$ satisfies the equation $C \cdot \varphi = \chi_\lambda(C) \varphi$.

There is a uniquely determined linear operator $D: \mathcal{H}_{|\mathbb{T}}^{(\sigma, \tau)} \rightarrow \mathcal{H}_{|\mathbb{T}}^{(\sigma, \tau)}$ (called *the radial part* of the Casimir operator C) such that on $\mathcal{H}^{(\sigma, \tau)}$ we have

$$|\mathbb{T} \circ C = D \circ |\mathbb{T}, \quad (3.7)$$

where the symbol C denotes the left action of the element $C \in \mathcal{U}_q$ on $\mathcal{H}^{(\sigma, \tau)} \subset \mathcal{A}_q$. It is possible to compute an explicit expression for the radial part D . Recall that with any spherical weight $\lambda = (\mu_1, \dots, \mu_l, 0, \dots, 0, -\mu_l, \dots, -\mu_1)$ in the weight lattice P one can naturally associate a dominant weight $\mu = (\mu_1, \dots, \mu_l)$ in the weight lattice P_Σ .

Theorem 3.3 — *The radial part*

$$D - \chi_\lambda(C) \text{id} : \mathbb{C}[x^{\pm 1}]^W \rightarrow \mathbb{C}[x^{\pm 1}]^W \quad (3.8)$$

is a constant multiple of Koornwinder's q -difference operator $D_{\varepsilon_1} - c_{\mu\mu} \text{id}$, defined as in (1.1) with base q^2 and parameters

$$a = -q^{\sigma+\tau+1}, \quad b = -q^{-\sigma-\tau+1}, \quad c = q^{\sigma-\tau+1}, \quad d = q^{-\sigma+\tau+2(n-2l)+1}, \quad t = q^2. \quad (3.9)$$

Note that the parameters (3.9) satisfy condition (1.3).

For any $\lambda \in P_\ell^+$, let us fix a non-zero element $\varphi(\lambda) \in \mathcal{H}^{(\sigma, \tau)}(\lambda)$ (zonal spherical function). Using Theorem (3.3) and the definition of the Askey-Wilson polynomials P_μ we can prove:

Theorem 3.4 — *The restriction $\varphi(\lambda)|_{|\mathbb{T}}$ of the zonal spherical function $\varphi(\lambda)$ ($\lambda \in P_\ell^+$) to the toral subgroup $\mathbb{T} \subset U_q(n)$ is equal to the Askey-Wilson polynomial*

$$P_\mu(x; -q^{\sigma+\tau+1}, -q^{-\sigma-\tau+1}, q^{\sigma-\tau+1}, q^{-\sigma+\tau+2(n-2l)+1}; q^2, q^2) \quad (3.10)$$

up to a scalar multiple.

Remark 3.5 — As stated in section 2, the subspaces $W(\lambda)$ are mutually orthogonal with respect to the inner product on \mathcal{A}_q defined in terms of the Haar functional. This means that the restricted zonal spherical functions $\varphi(\lambda)|_{|\mathbb{T}}$ are mutually orthogonal with respect to the inner product on the algebra $\mathcal{H}_{|\mathbb{T}}^{(\sigma, \tau)}$ induced by the restriction of the Haar functional.

Remark 3.6 — In the limit $\sigma \rightarrow \pm\infty$, the coideal \mathfrak{k}^σ tends to a two-sided coideal in \mathcal{U}_q which can be viewed as a direct q -analogue of the Lie subalgebra $\mathfrak{gl}(l) \oplus \mathfrak{gl}(n-l) \subset \mathfrak{gl}(n)$. The corresponding quantum Grassmannian has the usual Plancherel decomposition and can be regarded as the quotient of the quantum unitary group $U_q(n)$ by the quantum subgroup $U_q(l) \times U_q(n-l)$. The limit transition of our zonal spherical functions to this case is expected to be consistent with the limit transition (cf. [KS]) of general multivariable Askey-Wilson polynomials to multivariable big q -Jacobi or little q -Jacobi polynomials (cf. [ST1]).

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